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Quantifying the World and Its Webs: Mathematical Discrete vs Continua in Knowledge Construction

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Abstract

This short paper is meant to be an introduction to the 'Letter to Alan Turing' that follows it. It summarizes some basic ideas in information theory and very informally hints at their mathematical properties. In order to introduce Turing's two main theoretical contributions, in Theory of Computation and in Morphogenesis (an analysis of the dynamics of forms), the fundamental divide between discrete vs. continuous structures in mathematics is presented, as it is also a divide in his scientific life. The reader who is familiar with these notions, and is convinced that they (and their differences) are relevant in the mathematical understanding of phenomena, may skip this introduction and go directly to the Letter.

Keywords

Big Data, causality, coding, continua, discrete

As a mathematician, I will focus on the consequences on knowledge construction of the very 'mathematical structures' the new technologies of information are based on. The claim is that the use of discrete state (digital) devices, both as mathematical models and as a knowledge paradigm in science and humanities, is far from neutral. It will be then possible for the reader to develop some consequences of how the cultural and social relations may be affected by these technologies and their networks. In particular, these networks provide tools for knowledge as well as an *image* of the world; but, by their peculiar mathematical structure, the 'causal relations' of phenomena, in all areas of knowledge, are often redesigned according to the relations proposed by the digital networks and their internal causality. I will discuss these issues in the informal style

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of a 'personal letter' to Alan Turing. But let's first informally introduce some mathematical bases for this distant dialogue with the founding father of our computational universe.

Information can be elaborated and transmitted in many ways. In particular, control theory and the novel mathematics of the 'Geometry of Information' use tools from the mathematics of continua for these purposes.[1] However, these approaches are limited to specific applications and do not participate relevantly in the digital universe we are focusing on here.

'Digital' refers to the use of binary sequences to encode information. As we all know, two digits are enough to encode all integer numbers, thus all finite strings of symbols. On these grounds, Turing and Shannon invented two remarkable theories of *elaboration* (Turing, in 1936) and of *transmission* (Shannon, in 1948) of information. They both had predecessors, yet their work set the actual basis of the mathematical frame for the current technologies. The key aspect I want to focus on is the use of discrete mathematics in their work and its consequences – digital coding is just a form of coding of any discrete structure, in space and time.

But what does 'discrete' mean? The only sound way one can fully characterize discrete structures is mathematical: a structure is discrete when, intuitively, it is 'totally disconnected' or all its points are 'scattered'. Mathematically, a structure is discrete when the discrete topology is 'natural', that is, when all its points are *naturally* 'isolated'. What does 'natural' mean in mathematics? Well... it means that 'it works' or that it is pertinent ... to the intended applications, to physics, typically (see below). For example, if you give the discrete topology to Cantor's line of the real numbers (all points are open and closed, thus isolated), a continuum that everybody studies at school, this is not 'natural': all points are (artificially) isolated from each other, then all functions are continuous; there is no 'natural' notion of differentiation... and you can't do much interesting mathematics on it.

The key issue for us is that, in a topologically discrete structure, the access to its elements or points is exact, i.e. by measurement one knows them with certainty: in this sense all points are isolated with respect to the intended metrics, thus topology. In particular, in discrete, arithmetical structures, like the countable sequence of integer numbers or the computer's digits, all points are *naturally* isolated, thus accessible exactly – 0 or 1 and nothing else (we will discuss 'noise' below). This proposes an image and a practice of absolute certainty, of regularity and perfect iteration that modifies human knowledge, by affecting the deepest forms of our mathematical interaction with the (physical) universe. It contributes to an image of the world that erases centuries of knowledge construction in science: already Galileo hinted at the intrinsically approximated measurement in physics, which is always an interval. Arithmetical certainty brings us back to absolutes of thought that may forbid alternative thinking, since access and knowledge rules are exact and absolute, as in a digital computer: no nuances, no approximation, no uncertainty. Turing, by working both on digital machines and on continuous genesis of forms (see the 1952 paper that follows), will play with this 'fundamental aporia of mathematics', as Réné Thom called it: the discrete vs. the continuum, an issue at the core of the letter presented here.

Several immensely important mathematicians and physicists were aware of this. Bernhard Riemann (1826–1866), in his Habilitation (1854), set the basis of modern differential geometry and, in particular, of the mathematics of relativity theory. As for the discrete vs continuous spaces (manifolds), he observes: 'In the case of discrete manifolds, the comparison with regard to quantity is accomplished by counting, in the case of continuous manifolds by measuring'. And then a revolutionary remark, a 'divination', as H. Weyl, a great mathematician of relativity, called it, in 1921:

> in a discrete manifold, the ground of its metric relations is given in the notion of it, while in a continuous manifold, this ground must come from outside. Either therefore the reality which underlies space must form a discrete manifoldness, or we must seek the ground of its metric relations outside it, in binding forces which act upon it.[2]

In view of Riemann's work on the relation between space curvature and metrics, this opened the way to Einstein's approach, where gravitation (the binding force among physical bodies) is understood as inertial movement in spaces of varying curvature. The impact of relativity theory on 20th-century science and general culture is well known: it taught us a relativizing perspective – knowledge begins when you make explicit your reference system and its metrics and then analyse what is invariant (or not invariant) when changing the reference and/or its metrics. In particular, the *continuous* deformation of space-time modifies the metrics, i.e. the way phenomena are accessed and measured.

The alternatives in Riemann's two remarks are crucial: in discrete structures, one can only count and the metrics is intrinsic; in a *continuous* structure, one can measure (and count the number of measurements, of course) and the metrics is grounded on the 'binding forces' (gravitation, as Einstein will prove) by its relation to space curvature. So, in discrete structures, exit physical measurement, exit the understanding of gravitation as inertial movement in spaces of continuously changing curvature (and thus metrics) that revolutionized science: you can just count and work in an absolute metrics. Indeed, distances are fixed for good and the limitation to knowledge due to measurement is excluded. As observed since Galileo, classical measurement (but relativistic measurement as well) is always 'approximated', that is, it is always given as an *interval* in continua.[3] This is also why the 'interval topology' is called the 'natural topology' on the real numbers: its mathematical *naturality* derives for physical measurement, an interval, and provides a fundamental link between mathematics and physics, at least since Newton's and Leibniz's invention of differential calculus, in continua.

Shortly after Riemann, Henri Poincaré (1854–1912) showed how fluctuations below the best (and unavoidable) interval of measurement can yield totally unpredictable evolution also in very simple deterministic systems.[4] This further brought to light the role of measurement by intervals in continua. That is, he showed that '*des nuances presque insensibles*' (not measurable) could deeply and unpredictably affect classical dynamics by causing, over time, (large) measurable and unpredictable effects – 'and we have random phenomena', in fully deterministic systems, he observed. Turing's work on continuous morphogenesis (1952) will use these properties of non-linear deterministic dynamics: minor fluctuations, below measurement and inevitably diverse, trigger a rich variety of forms, whose detailed structure is highly unpredictable.

In short, on one side, the major epistemological teaching of relativity theory is that scientific knowledge begins when the observer actively chooses a reference system and a measure; then he or she analyses what is invariant (or not invariant) with respect to the *continuous transformations* of either or both of them. On the other, Poincaré showed the key role of approximated measurement in knowledge construction and, thus, of randomness in deterministic systems. Minor fluctuations – nuances, below measurement – contribute to determine the dynamics as they may cause measurable but unpredictable phenomena. Measurement as approximated with no a priori lower bound, that is, measurement as given by an interval in mathematical continua, was essential to this understanding. Note that it is the knowing subject who fixes the scale that defines what fluctuations and nuances mean. Both approaches enriched science immensely and moved us away from the myth of absolute universes and exact knowledge: scientific objectivity is the result of an active construction of invariants, modulo the inevitable approximation of our forms of access to the world (by measurement). The mathematical continuum turned out to be a fundamental tool for these scientific revolutions.

Also quantum mechanics is lost when organizing the world in a purely discrete manner: non-commutativity of measurement is given 'below Planck's h', in an interval. The (very surprising) discreteness in QM is found in the spectrum of energy of bound electrons. Yet, free electrons have a continuous spectrum and Schrödinger's equation is given in the continua of Hilbert's spaces. In short, the background spaces and time of QM are continuous, while those phenomena that break continuity (the energy spectrum, the spin-up/spin-down of a quanton, the entanglement effects...) are extremely puzzling for knowledge. As a major physicist, James Jean (1877–1946), observed: ‘when the discrete gets in, causality goes out’.[5]

As a matter of fact, so far, we understand causality only by continuous interactions and/or in continuous fields. We will perhaps do better in the future, but the (prevailing) interpretation of the discrete spin-up/spin-down of a quanton is given by a reference to pure contingency: it has no cause.[6] Similarly, quantum entanglement (the relativisticallly impossible correlation of topologically separable particles) is causally puzzling.[7]

In summary, within a discrete, arithmetical thinking of the universe, either we face deep challenges for knowledge, or we are left with absolute and exact access to the world (measurement, metrics) and no causality, as in computable/arithmetic structures. When a dynamics unfolds on your computer screen, the program describes rules that transform digits into digits: it is a ‘re- writing system’, where sequences of signs are replaced by others (i.e. re-written).[8] The visible dynamics, may it be the computational modeling of a falling stone or a moving crowd of humans, is formally implemented by the software as an exact, alpha-numeric, (re-)writing of signs, coded by the digits you see on the screen, where 0’s are replaced by 1’s (and vice versa) following a replacement rule: no ‘movement’ takes place, but just signs or pixels rewriting, which, moreover, may be iterated identically, as the signs are exact.[9] The deep physical-causal structure is totally hidden and irrelevant: this is a continuous flow of electricity in the hardware structure, which undergoes some critical transitions that locally produce a 0 or a 1. Its physical nature plays no role in the programmer’s design and the intelligibility of the transformations on the digital screen that unfold exactly, in the discrete. This is based on Turing’s fantastic idea of radically separating software, the programming or re-writing rules, from hardware, the electromagnetic or whatever support: the latter must only be a sufficiently stable support so as to produce exact discrete pixels, in spite of the fluctuations, perturbations, etc., of the underlying continuous flows – a non-obvious engineering achievement.

The role of discrete structures in the search for exactness and absolute certainty was also understood by another mathematician, David Hilbert (1862–1943). By proposing a formal approach to the foundation of mathematics, he wanted to secure for good the mathematical work. The latter could be done in continua, in large infinities, etc., yet, its *meta-mathematics* (the language and rules for doing mathematics) had to be based on arithmetizable languages and on formal deduction of a purely computational nature, that is, as re-writing systems of (possibly coded) signs, as will be noted later. In short, certainty for Hilbert had to be found in the ‘potential mechanizability’ of deduction, as formal replacement rules, in purely formal languages (no reference to ‘meaning’). As for Frege, arithmetic had to be the bottom line of certainty in the foundation of mathematics. This was supposed to yield the 'definitive elimination of the problem of the foundation of mathematics', he observed in 1927. Again, a quest for exact knowledge, absolute certainty, thus a final solution of all foundational and meaning problems by the exact unfolding of signs in discrete space and time structures. Fortunately, there are no final solutions in science. Yet, this approach to knowledge by formal logic, in the absolute, by discrete space of meaningless sequences of signs, is still part of the search for certainty in humanities: in many areas, the computational approaches also derive their legitimacy from this perspective. As I have said, Hilbert proposed the formal approach at the level of the *language* in which we talk of these mathematics that had brought to attention the 'delirium' (Frege, 1960 [1884]) of continually curbing spaces (Riemann, 1854) and, later, the key role of the interval of measurement in continua (Poincaré, 1892), quantum indetermination (1900), which are at the core of today's intelligibility of the physical world. That is, Hilbert proposed a *meta*-mathematical investigation, in order to save, by a foundation on the certainty of axioms and mechanical rules, the modern applications of mathematics to physics – which could be as audacious as needed. When algorithms, as mechanizable sets of rules, are still now proposed to solve all problems in an area of science or of humanities, Hilbert's program is abusively extended to those areas, in a search for the elimination of the knowing subject in knowledge construction, his or her relativizing perspective and the approximation of access due to measurement

Gödel, Church and Turing, in different ways, as I will mention in my letter to Turing, set a halt to this frightening project of the definitive elimination of the foundational uncertainty in meta-mathematics. As a consequence, openness of access and meaning of natural phenomena cannot be expelled from the applications of mathematics either, as I will discuss with regard to Turing. In order to do this, they had to define exactly the meaning of mechanizable or computable. So, they mathematically defined formal languages and machines for computing and showed their deductive and computational incompleteness: there is no final solution to all well-defined problems. Today, we know of very interesting (concrete) purely arithmetical problems that cannot be solved within formal arithmetic.[10] Yet, while setting the limits of computation, the system proposed by Turing set the basis for modern computing, including the fundamental notions of software, as distinct from hardware, and of operating system and compiler, by his 'universal machine'.

As mentioned above and further discussed in the 'letter' to follow, Turing, in his short life (1912–1954), crossed the border of this fundamental divide, discrete vs continua. He first invented the Logical Computing Machine (1936), a purely arithmetical device for deducing and . . . disproved Hilbert's program. He then stressed its discrete nature by calling it, after the Second World War, a Discrete State Machine (DSM). This happened at a time when, not focusing on logic anymore, he addressed natural phenomena and wrote a major paper on 'continuous dynamics of forms', that is, on morphogenesis (1952). This passage, from discrete to continua, will allow us to stress the relevance of Turing's distinction between *imitation* and *modeling*, as ways to approach phenomena.

In summary, the tools from discrete mathematics are far from neutral in addressing knowledge, both in science and in humanities. The use of these tools for analyzing the language of mathematics (Hilbert's program) consistently originated from the dream (or nightmare) of eliminating from the foundation of science the problems raised by the novel relevance of spaces of varying curvature, approximation and uncertainty of measurement.[11] It was then transferred to general knowledge construction, when digital computers became an integral part of human interaction. The writing of an algorithm and its correctness aims at predictability and certainty: a machine must follow the rules; divergence from these rules is 'noise' or 'bugs' to be eliminated. This is program correctness. Discrete databases are exact and allow only counting; their metrics is intrinsic: no approximation of access, no nuances. A computer *iterates identically* the implementation of the wildest turbulence, when pushing the 'restart' button, as the initial and contour conditions are given exactly, by digits. This is physically absurd or just a (often very relevant) 'imitation' of reality, not a 'model' addressing intelligibility, as I will stress in the letter to Turing, who invented this distinction. This imitation and the digital/discrete view of the world becomes a model of or is even identified with reality itself, the programmer and the user who deal with digital machines, which iterates exactly. Of course, noise and bugs are always possible, but all is done to avoid them, and it works: there are very few of them, even in networks of computers where the continua of space and time (and fluctuations of all sorts) introduce the uncertainties I mentioned above. The discrete nature of the nodes, the servers and computers, in the worldwide web, say, allows the miracle of great reliability as certainty and predictability of network programming. Thus, computer networks and databases, if considered as an ultimate tool for knowledge or as an image of the world, live in the nightmare of exact knowledge by pure counting, of unshakable certainty by exact iteration, and, still today for many, of a 'final solution' of all scientific problems (the Big Data approach; see the letter to Turing).

Our culture and science are highly affected by this: 'the universe is a (huge) Turing machine', says Wolfram[12] against Turing, as I will recall in the letter. Along this line, 'the brain is a digital computer', say many others. They neglect by this the fundamental dualism invented by Turing (hardware vs software), as radical as Descartes' and of which Turing was fully aware. 'The DNA is a program', claim others and disregard morphogenetic effects in ontogenesis, such as those described by Turing in continua: the genocentric myths in biology are largely based on the idea of the completeness of the formal rules and data inscribed in the DNA. And this computational frame in biology made us blind, for more than 50 years, by the myth of exact macromolecular cascades (‘specificity’), to endocrine disruptors and carcinogens that affect in probabilities the largely stochastic macromolecular interactions, in the continua of their enthalpic oscillations both in hormones’ pathways and proteomes’ turbulences.[13]

In conclusion, the networks of a digital computer are far from neutral with regard to the image of the world they explicitly and implicitly present. They force a fake objectivity by their underlying mathematical structure, with its intrinsic metrics and exact access: all is arithmetical counting, which is an absolute. This excludes the fundamental interfaces where subjectivity plays a key role in knowledge construction: the ‘relativizing’ choice of a reference system and its invariant properties (0 is an absolute origin of counting); the dependence of the metrics on the context, as Riemann had observed; approximated measurement, by an interval – an active human gesture; it excludes uncertainty, nuances; the radical, non-dualistic, materiality of biological organisms and its complex blend of continuous and discrete dynamics.

Of course, trained scientists use in fantastic ways computational tools for modeling. Indeed, most contemporary science would not exist without computational modeling and databases. Yet, the awareness of the often implicit biases imposed by the underlying discrete mathematics is required to implement and understand computational activities at their best. Unfortunately, as for common sense knowledge, the computational bias is omni-present at a ‘subconscious’ level and it shapes imagination and action. The highest and most recent point of the computational distortion of knowledge construction, its alienation towards exact machine computations, is proposed by the views of Big Data technologies without science.[14] These views pretend to replace the complex interplay between knowing subject and accessible world, thus the knowing subject and his or her theoretical debatable proposals and measurements, by data mining in immense databases. This would allow prediction and action with no need of understanding. I will present below this novelty to Turing, as he would surely appreciate the need we have to extend the scientific methods by the inclusion of counting on the fantastic databases we have today, not to bypass them. This requires the construction of a new knowing subject who integrates these tools in an enlarged scientific rationality: beyond Greek observation and theoretizing and Galileo’s experimental methods, we can also collect, compare and use today an immense amount of digitalized information, for enhancing knowledge. This may help in seeing and proposing relations where there seem to be none, in conjecturing new theories, in correlating meanings. Against the myth of a final solution of knowledge problems by pure data mining, I will mention to Turing some recent limitative results I (co-)worked at developing,[15] which mathematically prove that 'computational brute force' in databases bumps into a deluge of spurious correlations, which do not allow either prediction or action. This shows the need for a knowing subject, with his or her choice of (pertinent) observables and form of access and measurement, his or her concrete friction to reality and knowledge proposal, in order to construct objectivity, possibly also by a scientific use of computers' nets and data mining.

Notes

1. See Levine (1996). A seminal paper on continuous elaboration of information is Bush (1929). As for Geometry of Information, an entire scientific community working on it may be found at: http://forum.cs-dc.org/category/72/geometric-science-of-information.
2. See Bernhard (1854: 14, 36–7).
3. These intervals may be arbitrarily reduced in length as, in principle, one cannot assume an a priori and fully general lower bound to measurement. This is how Cantor and Dedekind constructed a remarkable mathematical continuum, by letting all intervals converge to a point, at the mathematical limit.
4. See Poincaré (1892).
5. Some dramatic consequences of this 'linguistic approach' to biology are presented in Longo (2018; see https://www.di.ens.fr/users/longo/download.html).
6. 'Hidden variable theories' search for hidden causes of all quantum events in underlying continua. I strongly prefer the a-causal interpretation, as I do not see how any event should have an original cause in the Big Bang.
7. See Le Bellac (2013).
8. See Bezem et al. (2003).
9. Possibly, just sequences of 0's or 1's.
10. See Longo (2011).
11. Some philosophers of logic claim that the foundational crisis in mathematics was due to a minor 1901 game play concerning barbers unable to shave themselves. Hilbert's fundamental text on foundations dates from 1899, where he proposes the analytic encoding of all geometries (Riemannian) in arithmetic. In 1900, at the Paris Conference, he raised the problem of a finitistic proof of the consistency of arithmetic as a solution to the foundational crisis due to the invention of non-Euclidean geometries.
12. See Wolfram (2013).
13. See Longo (2018).
14. See Anderson (2008).
15. Calude and Longo (2016). See also Longo's 'Letter to Alan Turing' at: https://www.di.ens.fr/users/longo/download.html.

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