Yakir Aharonov and Daniel Rohrlich

Quantum Paradoxes

Quantum Theory for the Perplexed

WILEY-VCH

WILEY-VCH Verlag GmbH & Co. KGaA

This Page Intentionally Left Blank

Y. Aharonov and D. Rohrlich

Quantum Paradoxes

Quantum Theory for the Perplexed

This Page Intentionally Left Blank

Yakir Aharonov and Daniel Rohrlich

Quantum Paradoxes

Quantum Theory for the Perplexed

WILEY-VCH

WILEY-VCH Verlag GmbH & Co. KGaA

Authors

Prof. Yakir Aharonov
School of Physics and Astronomy
Tel Aviv University
Ramat Aviv 69978
ISRAEL

and

Department of Physics
University of South Carolina
Columbia SC 29208
U.S.A.

Dr. Daniel Rohrlich
Department of Condensed Matter Physics
Weizmann Institute of Science
Rehovot 76100
ISRAEL

Cover Picture
© Charles Addams, Courtesy Tee and Charles Addams Foundation

Library of Congress Card No.: applied for
British Library Cataloging-in-Publication Data:
A catalogue record for this book is available from the British Library

Bibliographic information published by Die Deutsche Bibliothek
Die Deutsche Bibliothek lists this publication in the Deutsche Nationalbibliografie; detailed bibliographic data is available in the Internet at <http://dnb.ddb.de>.

Printed in the Federal Republic of Germany
Printed on acid-free paper

Composition Steingräber Satztechnik GmbH, Dossenheim
Printing Strauss GmbH, Mörlenbach
Bookbinding Litges & Dopf Buchbinderei GmbH, Heppenheim

ISBN-13: 978-3-527-40391-2
ISBN-10: 3-527-40391-4

Preface

Quantum Paradoxes is a series of studies in quantum theory. Each chapter begins with a paradox motivating the study, in the rest of the chapter, of a fundamental aspect of the theory. We hope that this style in physics – progress through paradox – will rub off on readers. The studies, taken together, set out a new interpretation of quantum theory.

Elements of this interpretation include topological phases (the Aharonov-Bohm effect and its generalizations), "modular" variables, nonlocal measurements and relativistic causality, time-symmetric boundary conditions, measurement of the quantum wave, "weak" measurements and "weak" values, and new axioms for quantum theory. A treatment of "quantum measurements", starting in Chap. 7, plays an important role in the book. Indeed, measurement is so important that many of the works cited in the book can be found in the anthology *Quantum Theory and Measurement*, edited by J. A. Wheeler and W. H. Zurek (Princeton: Princeton U. Press), 1983; these citations include a note "reprinted in WZ" with page numbers.

For whom is this book written? It is designed for physics students, physicists and philosophers of science with an interest in fundamental aspects of quantum theory. The first two chapters of *Quantum Paradoxes* do not require prior knowledge of quantum theory, and Chaps. 3–4 introduce basic notions of states, observables and quantum phases, so students can use the book even during a first course in quantum mechanics. It is not, however, a substitute for such a course.

Each chapter ends with a problem set. Problems marked with an asterisk (*) are, in general, less straightforward than others.

It is a pleasure to thank those who have helped us write this book. We are indebted to colleagues (including students) who read parts of the book at one stage or another, most especially Philip Pearle and Fritz Rohrlich, and to Elisabeth Warschawski for much encouragement and technical support. We thank Shula Volk for opening her ceramics studio to us at odd hours of the day and night, as reported in Sect. 2.1. We also acknowledge support from the Giladi Program of the Israeli Ministry of Absorption and from the Ticho Fund.

Yakir Aharonov
Daniel Rohrlich
December 2004

Quantum Paradoxes: Quantum Theory for the Perplexed. Y. Aharonov and D. Rohrlich

ISBN 3-527-40391-4

This Page Intentionally Left Blank

Contents

1 The Uses of Paradox

On November 9, 1919, *The New York Times* reported solar eclipse observations confirming a prediction of Einstein's general theory of relativity: rays of starlight bend near the sun. It also reported that when Einstein sent his theory to the publishers, "he warned them that there were not more than twelve persons in the world who would understand it ...." Was there a time when only "twelve wise men" understood the general theory of relativity? "I do not believe there ever was such a time," commented Feynman. "There might have been a time when only one man did, because he was the only guy who caught on, before he wrote his paper. But after people read the paper a lot of people understood the theory of relativity in some way or other, certainly more than twelve. On the other hand, I think I can safely say that nobody understands quantum mechanics." [1]

What is the problem with quantum mechanics? It is a spectacularly successful theory. It governs the structure of all matter. Measurements of Planck's constant are accurate to better than a part in a million, and still more accurate measurements confirm predictions of quantum electrodynamics. But along with the spectacular successes of quantum mechanics come spectacular difficulties of interpretation. "Do not keep saying to yourself, if you can possibly avoid it, 'But how can it be like that?"' Feynman continued, "because you will get 'down the drain', into a blind alley from which nobody has yet escaped. Nobody knows how it can be like that."

We can stop asking ourselves, "But how can it be like that?" We may indeed despair of asking a question that Einstein, Schrödinger and Feynman could not answer. But we cannot stop *using* quantum mechanics. So the problem is that everybody uses quantum mechanics and nobody knows how it can be like that. Our relationship with quantum mechanics recalls a Woody Allen joke:

> This guy goes to a psychiatrist and says, "Doc, my brother's crazy – he thinks he's a chicken! And, uh, the doctor says, "Well, why don't you turn him in?" And the guy says, "I would, but I need the eggs!"

We say, "Quantum mechanics is crazy – but we need the eggs!"

Such a relationship with quantum mechanics is paradoxical. In this book, we will not be satisfied to have a paradoxical relationship with quantum mechanics. We will not stop asking, "How can it be like that?" But we will *use* paradox repeatedly in order to understand quantum mechanics better.

Quantum Paradoxes: Quantum Theory for the Perplexed. Y. Aharonov and D. Rohrlich

ISBN 3-527-40391-4

1.1 Paradox in Physics

We will use paradox to probe quantum mechanics. Can paradox be useful? The history of physics shows *how* useful. As Wheeler [2] put it, “No progress without a paradox!” In this section, we define and classify physics paradoxes; the next sections present examples of each class.

A paradox is an argument that starts with apparently acceptable assumptions and leads by apparently valid deductions to an apparent contradiction. Since logic admits no contradictions, either the apparently acceptable assumptions are not acceptable, or the apparently valid deductions are not valid, or the apparent contradiction is not a contradiction. A paradox is useful because it can show that something is wrong even when everything appears to be right. It does not show *what* is wrong. But something is wrong – something we thought we understood – and a paradox moves us to reexamine the argument until we find out what is wrong.

We can classify physics paradoxes according to what is wrong. There are three broad classes: “errors”, “gaps” and “contradictions”.

Many paradoxes arise from errors. An error in logic or in our understanding of the laws of physics easily leads us to an apparent contradiction. Our error may be simple or it may be subtle, but it is just an error; once we recognize it, we have resolved the paradox. What distinguishes the first class is that these paradoxes do not arise from any flaw in the theory. In the special theory of relativity, for example, erroneous assumptions about simultaneity lead us to paradox. (See Sect. 1.2.) Resolving the paradox, we improve our understanding of special relativity, but we do not improve the theory. Another example of a paradox arising from an error is Einstein’s clock-in-the-box paradox. (See Sect. 2.4.) Einstein made an error and arrived at an apparent contradiction in quantum theory. The resolution of the paradox came as a surprise, but it did not show quantum theory to be flawed in any way.

Other paradoxes *do* show a physical theory to be flawed. A gap in physical theory is a flaw. As an example of a gap, consider Wheeler’s paradox of black hole entropy. According to the general theory of relativity, nothing can escape a black hole. We, as outside observers, can measure the electric and gravitational fields of a black hole, and hence its charge and mass (and angular momentum); but we have no other access to a black hole. So a black hole at rest has only three properties: charge, mass and angular momentum. Such a simple physical system can hardly have much entropy. Now suppose a complicated physical system, containing a lot of entropy, falls into a black hole. What happens to the entropy? Apparently it vanishes. But vanishing entropy violates the second law of thermodynamics. Wheeler told his student Bekenstein about this paradox:

> The idea that a black hole had no entropy troubled me, but I didn’t see any escape from this conclusion. In a joking mood one day in my office, I remarked to Jacob Bekenstein that I always feel like a criminal when I put a cup of hot tea next to a glass of iced tea and then let the two come to a common temperature, conserving the world’s energy but increasing the world’s entropy. My crime, I said to Jacob, echoes down to the end of time, for there is no way to erase or undo it. But let a black hole swim by and let me drop the hot tea and the cold tea into it. Then is not all evidence of my crime erased forever? This remark was all that Jacob needed [3].

Bekenstein [4] proposed that a black hole has entropy proportional to the square of its mass. If any physical system falls into a black hole, the mass of the black hole increases – and hence the entropy. He demonstrated that the increase in entropy is at least as great as the entropy of the infalling system, thus corroborating the second law and resolving Wheeler's paradox.

Wheeler's paradox indicated a flaw – but not a fatal flaw – in general relativity and thermodynamics. The resolution of the paradox did not invalidate either theory. The apparent contradiction between the theories arose from a gap in thermodynamics – we didn't know how to extend the concept of entropy to black holes – and Bekenstein's proposal filled the conceptual gap. Another paradox in the second class came, in turn, from Bekenstein's proposal: if thermodynamics extends to black holes, then black holes must emit as well as absorb heat. But nothing can escape a black hole! This paradox, too, arose from a conceptual gap, as Hawking discovered: one consequence of the uncertainty principle is that black holes *radiate* [5]. Many such paradoxes appear in this book.

A contradiction in physical theory is a fatal flaw. Paradoxes in the third class are associated with revolutions in physics, because they indicate that the physical theory behind the paradox is wrong. Bohr faced such a paradox in 1911. In that year, Rutherford reported experiments on neutral atoms, showing that the positive charges in atoms – but not the negative charges (electrons) – are concentrated in nuclei. According to classical theory, such atoms should be unstable: like all accelerating charges, the electrons should radiate energy, and fall into the nuclei. Matter should collapse in a split second. So why is matter stable? Bohr realized that this paradox had no resolution in classical physics. Only a new physical theory – quantum theory – could resolve it. The only resolution was a revolution.

The paradox arose for Bohr as a contradiction between physical theory and experiment. Especially useful are paradoxes that arise as contradictions *within* physical theory. Such a paradox can show that a physical theory is wrong even when no experiment contradicts it. The paradox then starts us searching for a new theory. (See Sects. 1.4 and 2.2.)

1.2 Errors

Every student of special relativity encounters the Twin Paradox [6]. Here is a Triplet Paradox. Dumpy, Grumpy and Jump – identical triplets wearing synchronized wristwatches – once lived together happily at home. But Grumpy got mad at Dumpy and decided to move to another city. When he arrived, his watch was still synchronized with his brothers' watches, because he travelled very slowly compared to the speed of light. (In this paradox we set the speed of light to 1000 m/s.)

A month later, Jump decided to visit Grumpy. Dumpy accompanied Jump to the train station, and Jump took a seat in the train. Then the train accelerated, within a second, to 100 m/s. At the end of this second, Jump's cabin passed Dumpy on the platform. Jump and Dumpy glanced at each other through a cabin window and noticed that their watches still showed the same time (to within a second). Hence Jump did not age appreciably during the acceleration. For the rest of the trip, the train's speed and direction were constant. When it arrived, it stopped within a second.

Dumpy and Grumpy expected that Jump would be slightly younger than them when he arrived, and that his watch would lag behind their watches, for Jump had been moving fast relative to them. But Jump expected the opposite: Dumpy and Grumpy would be slightly younger, and their watches would lag behind his. He told himself, "After one second of acceleration, Dumpy's watch and mine showed the same time (to within a second); and Dumpy's and Grumpy's watches were still synchronized. Afterwards, the inertial reference frame of Dumpy and Grumpy moved fast relative to mine; so time passed more slowly for them, and their watches now lag behind mine." When he arrived, he discovered that his watch lagged Grumpy's by about half a minute! On the one hand, Jump's expectation should be just as correct as that of his brothers; there can be no preferred frame in special relativity. On the other hand, Jump and his brothers cannot all be correct. So special relativity contradicts itself!

The Triplet Paradox belongs to the class of errors in that it does not arise from any flaw or misconception in the special theory of relativity. It arises, rather, from incorrect intuition. We can often use paradoxes in this class to improve our intuition.

1.3 Gaps

In 1856, Clausius stated the second law of thermodynamics as follows: heat cannot flow from a colder body to a hotter body without an accompanying process (i.e. work). Clausius regarded the second law as exact, and tried to derive it from the laws of mechanics. In 1871 he published a paper [7] in which he offered a mechanical explanation of the second law. He did not know that Boltzmann had published much the same explanation five years earlier [8]. Boltzmann (who, like Clausius, regarded the second law as exact) was quick to claim priority [9]. Yet Clausius did not wholly concede [10]. Maxwell was amused. "But it is rare sport to see those learned Germans contending for the priority of the discovery that the 2nd law of $\theta\Delta cs$ is the Hamiltonische Princip . . . ." he wrote. "The Hamiltonische Princip, the while, soars along in a region unvexed by statistical considerations . . . ." [11]. Boltzmann and Clausius were both wrong. The second law has no mechanical explanation; it is statistical.

What made Maxwell so sure that the second law is statistical? In 1859 he had calculated that the distribution of molecular speeds in any gas, hot or cold, would range from zero to infinity. (Molecules were still an untested hypothesis at the time.) In 1867 he had considered the following thought experiment. Gas fills a sealed, insulated box, divided by a diaphragm. The gas is hot on one side of the diaphragm and cold on the other side; yet there are fast molecules in the cold gas and slow molecules in the hot gas. "Now conceive a finite being who knows the paths and velocities of all the molecules by simple inspection but who can do no work except open and close a hole in the diaphragm by means of a slide without mass." The being opens and closes the hole in such a way that fast molecules in the cold gas enter the hot gas and slow molecules in the hot gas enter the cold gas. Energy gradually flows from the cold gas to the hot gas. After many molecules have crossed through the hole, "the hot system has got hotter and the cold colder and yet no work has been done, only the intelligence of a very observant and neat-fingered being has been employed" [12]. The "neat-fingered being" soon had a name: "Maxwell's demon".

Maxwell's demon violates the second law of thermodynamics, as formulated by Clausius: it does no work, yet it causes heat to flow from a cold gas to a hot gas. It does not, however, violate the laws of mechanics. Hence the second law cannot be a mechanical law. Maxwell's

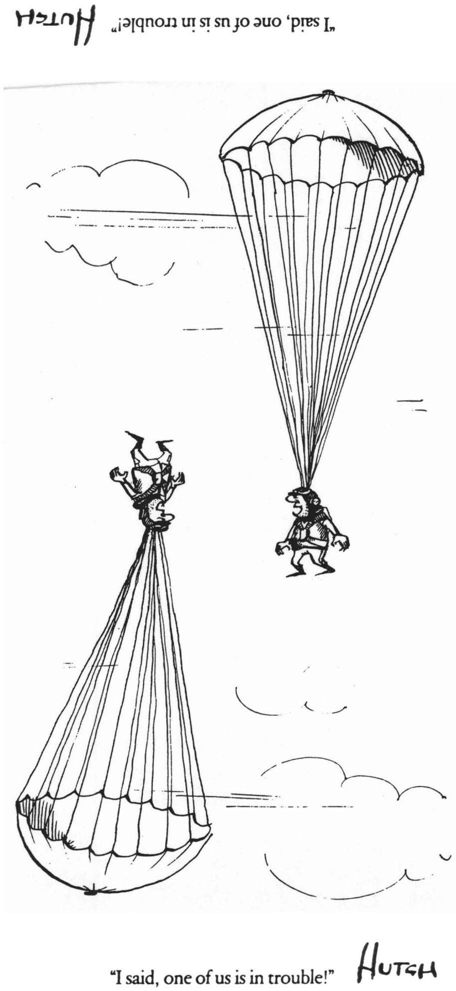

Figure 1.1: Two opposite “arrows of space”. [With thanks to Stuart M. Hutchison and Princeton Tiger Magazine.]

thought experiment was a paradox for Clausius’s formulation; it does not disprove the second law, but it shows that the second law can only be a statistical law.

Another formulation of the second law states that the entropy of a closed system always tends to increase to thermal equilibrium. But this formulation, too, leads to a paradox. It assumes an arrow of time, relative to which entropy tends to increase. But what if there is no arrow of time? What if the “arrow of time” is no more intrinsic than the “arrow of space” defined by gravity? (See Fig. 1.1.) Suppose that two sealed, insulated boxes are filled with gas, e.g. helium in one box and neon in the other, and at time $t = 0$, neither gas is at thermal equilibrium. Now on the one hand, if the boxes are perfectly insulated, they could contain two opposite arrows of time. Assume that the gases have contrary evolutions: the entropy of the neon increases in time while the entropy of the helium *decreases* in (the same) time. Such an assumption is plausible since the laws of mechanics are invariant under time reversal and the

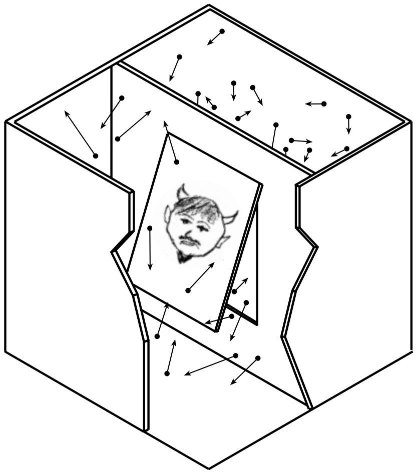

Figure 1.2: Maxwell's demon as a trapdoor.

boxes do not interact. On the other hand, suppose the boxes *do* interact, with an interaction that is independent of time; assume that the position and momentum of each atom at $t = 0$ is the same as before. According to the second law, the combined entropy of the two gases always tends to increase; that is, any perturbation of the helium atoms, however small, will destroy the precise coordination of their positions and momenta that allows their entropy to decrease. So in the evolution of the two gases after $t = 0$, their total entropy increases. But the same reasoning applies in reverse to the evolution of the gases before $t = 0$: their total entropy must *decrease* until $t = 0$. Extrapolation forwards from $t = 0$ implies that the neon (with its increasing entropy) overwhelms the helium; extrapolation backwards from $t = 0$ implies that the helium overwhelms the neon. This paradox shows that the second law contains no arrow of time. (See also Chap. 10.)

The second law is *almost* exact, i.e. the probability of a significant violation is very small. Maxwell's demon can violate the second law, yet the probability of a significant violation is very small. Still, after Maxwell, the demon turned up in new paradoxes. The demon kept turning up, because it is easier to imagine a demon that can violate the second law significantly, than to prove that it can't. For example, in Fig. 1.2 the demon is a trapdoor that apparently allows only fast molecules of the cold gas to enter the hot gas. In 1914, Smoluchowski showed that this demon fails to violate the second law significantly because the trapdoor itself thermalizes, eventually opening and closing in random fluctuations [13]. More recent paradoxes allow Maxwell's demon to measure and compute. Their resolution involves an application of information theory to thermodynamics [14].

All the paradoxes in this section belong to the class of gaps; they show up flaws or gaps in how we understand the second law, but do not invalidate it. The resolutions of these paradoxes correct our formulation of the second law and extend the concepts we use to apply it, but do not contradict the formalism of thermodynamics.

1.4 Contradictions

Maxwell's equations imply that a changing electromagnetic field in empty space propagates as a wave with constant speed c. On the face of it, this implication contradicts Newton's mechanics. According to Newton, if we run after a light wave, its speed (relative to us) decreases. Velocities add as vectors: if the velocity of a light ray with respect to Alice is $\mathbf{v}_A$ and the (nonzero) velocity of Alice with respect to Bob is $\mathbf{v}_{AB}$, then the velocity of the ray with respect to Bob is $\mathbf{v}_A + \mathbf{v}_{AB}$. The ray cannot have the same speed for Alice and for Bob. What, then, corresponds to c? Physicists of Maxwell's time assumed that electromagnetic waves propagate through a medium, the "aether", and what corresponds to c is the speed of the wave relative to the aether.

At first, the aether was a plausible assumption. Even before Maxwell, physicists assumed that light propagates through an aether. Every wave known to them, from ripples in water to sound in air, propagated through some medium. Fresnel showed in 1818 that an aether at absolute rest, unaffected by the earth's motion through it, would be consistent with the "aberration effect", a seasonal shift in the apparent positions of stars in the sky. Over the rest of nineteenth century, however, the aether became less and less plausible. In 1887, Michelson and Morley measured the speed of light parallel and perpendicular to the earth's motion, and found no difference.[1] Hence the aether and the earth must move together. Or else the earth is at absolute rest – Copernicus was wrong after all!

Aether was a paradox. But since an aether at absolute rest made sense of c (and defined the "absolute space" that Newton had postulated), most physicists chose tacitly to live with it. They then had to explain the contradictory experiments.

For Einstein, the paradox was different. He concluded early on (even without the Michelson–Morley experiment) that there *is* no aether. He was then left with the contradiction between Newton's mechanics and electromagnetism. At age 16, Einstein formulated the paradox as follows:

> If I pursue a beam of light with the velocity c (velocity of light in a vacuum), I should observe such a beam of light as an electromagnetic field at rest though spatially oscillating. There seems to be no such thing, however, neither on the basis of experience nor according to Maxwell's equations. From the very beginning it appeared to me intuitively clear that, judged from the standpoint of such an observer, everything would have to happen according to the same laws as for an observer who, relative to the earth, was at rest. For how should the first observer know, or be able to determine, that he is in a state of fast uniform motion? [15]

Newton proved that his laws of mechanics are the same for all observers in uniform (rectilinear) motion; and Maxwell [16] realized that – apart from the aether – electromagnetism is the same for all observers in uniform motion. If there is no aether, then the laws of mechanics and electromagnetism must be the same for all observers in uniform motion. The paradox was that in Newton's mechanics, which relates such observers by Galilean transformations, the speed of light is not a constant; in Maxwell's electromagnetism, which relates such observers by Lorentz

[1] To show that the basement laboratory of Michelson and Morley did not trap aether, Morley and Miller later repeated the measurements on a hilltop.

transformations, the speed of light *is* a constant, as it is in experiment. Years later, Einstein resolved the paradox by modifying Newton's mechanics so that Lorentz transformations, rather than Galilean transformations, relate observers in uniform motion.

Indeed, paradox can be useful but, as this example shows, the paradox has to be the *right* paradox. Where other physicists saw a contradiction between physical theory and experiment, Einstein (and to an extent Poincaré) saw a contradiction within physical theory. What is striking in this example is how ready Einstein was to discard the aether assumption when it had become implausible (but still accepted by all other physicists) and to face a fundamental contradiction in physical theory. He was thus able to identify the right paradox behind the wrong paradox, and later to resolve it.

1.5 Overview of the Book

In this book we have two goals. Our primary goal is a deeper understanding of all fundamental aspects of quantum mechanics. The first chapters do not assume prior knowledge of quantum mechanics. Chapter 2 discusses uncertainty and consistency in quantum theory, without the formalism, and Chap. 3 introduces the notion of a quantum state, while discussing the completeness of quantum theory. Chapter 4 introduces the quantum phase and probes its unique role in the theory with the help of Schrödinger's equation. Chapters 5 and 6 apply the phase in unconventional ways and raise unconventional questions about quantum mechanics. Chapters 7–11 present a theory of quantum measurements, a powerful tool for probing quantum mechanics. Chapters 12–13 explore connections among the Feynman path integral, Berry's phase and the Aharonov–Bohm and Aharonov–Casher effects. Quantum mechanics is nonrelativistic throughout Chaps. 1–13, but Chaps. 14–17 discuss relativistic quantum measurements, measurements of the quantum wave, and "weak" measurements within a new formalism adapted to relativity. Chapter 18 proposes simple physical axioms for quantum theory.

Our secondary goal is to encourage physicists to use paradoxes creatively, both in teaching and in research. We use paradoxes all through the book. Each chapter (except this one) begins with a paradox that motivates the rest of the chapter. Try to resolve the paradoxes as you read! Chapter 2 begins with a paradox from the class of contradictions; Chapter 10 begins with a paradox from the class of errors. Chapters 3–8 and Chaps. 11–18 all begin with paradoxes from the class of gaps; and it is an open question to which class the paradox in Chap. 9 belongs.

Problems

1.1 If Wheeler's paradox of Sect. 1.1 had preceded the discovery of quantum mechanics, to which class of paradoxes would it belong?

1.2 (a) Resolve the Triplet Paradox of Sect. 1.2. What was Jump's error?
(b) About how long did Jump's train ride last?

References

[1] R. P. Feynman, *The Character of Physical Law* (Cambridge, MA: M. I. T. Press), 1965, p. 129.

[2] J. A. Wheeler, *Trans. N. Y. Acad. Sci.* **33** (1971) 745; cf. p. 778.

[3] J. A. Wheeler with K. Ford, *Geons, Black Holes and Quantum Foam* (New York: W. W. Norton and Co.), 1998, pp. 312–15.

[4] J. D. Bekenstein, *Phys. Rev.* **D7** (1973) 2333.

[5] S. W. Hawking, *Comm. Math. Phys.* **43** (1975) 199.

[6] See, for example, E. F. Taylor and J. A. Wheeler, *Spacetime Physics*, Second Edition (New York: W. H. Freeman and Co.), 1992, pp. 125–26.

[7] R. Clausius, "On the reduction of the second law of thermodynamics to general mechanical principles" [in German], *Pogg. Ann.* **142** (1871) 433; cited in M. J. Klein, *Amer. Sci.* **58** (1970) 84.

[8] L. Boltzmann, "On the mechanical meaning of the second law of thermodynamics" [in German], *Wiener Berichte* **53** (1866) 195; cited in Klein, *op. cit.*

[9] L. Boltzmann, "On priority in discovering the relation between the second law of thermodynamics and the principle of least action" [in German], *Pogg. Ann.* **143** (1871) 211; cited in Klein, *op. cit.*

[10] R. Clausius, "Remarks on Mr. Boltzmann's priority claim" [in German], *Pogg. Ann.* **144** (1871) 265; cited in Klein, *op. cit.*

[11] J. C. Maxwell, letter to P. G. Tait, 1 December 1873, in C. G. Knott, *Life and Scientific Work of Peter Guthrie Tait* (Cambridge: Cambridge U. Press), 1911, pp. 114–16; cf. pp. 115–16.

[12] J. C. Maxwell, letter to P. G. Tait, 11 December 1867, in Knott, *op. cit.*, pp. 213–14; cf. p. 214.

[13] M. Smoluchowski, in *Vorträge über die kinetische Theorie der Materie und der Elektrizität* (Leipzig: Teubner), 1914, pp. 87–121. See also R. Feynman, R. Leighton and M. Sands, *The Feynman Lectures on Physics*, Vol. **I** (Reading, MA: Addison–Wesley Pub. Co.) 1963, Chap. 46.

[14] L. Szilard, *Behav. Sci.* **9** (1964) 301 [trans. of L. Szilard, *Z. Phys.* **83** (1929) 840], reprinted in WZ pp. 539–548; L. Brillouin, *J. Appl. Phys.* **22** (1951) 334; and C. H. Bennett, *Int. J. Theor. Phys.* **21** (1982) 905; all reprinted in H. S. Leff and A. F. Rex, *Maxwell's Demon: Entropy, Information, Computing* (Princeton: Princeton U. Press), 1990.

[15] A. Einstein, *Autobiographical Notes*, trans. and ed. P. A. Schilpp (Chicago: Open Court Pub. Co.), 1979, pp. 49, 51.

[16] M. Goldman, *The Demon in the Aether* (Edinburgh: P. Harris Pub.), 1983, pp. 202–3; J. C. Maxwell, *A Treatise on Electricity and Magnetism*, Vol. **II** (New York: Dover), 1954, pp. 241–43 (Articles 600 and 601).

This Page Intentionally Left Blank

2 How to Weigh a Quantum

Now we start to use paradoxes to investigate quantum theory and its mathematical formalism, quantum mechanics. We need the mathematical formalism, but we do not need it yet. We first take up the question: Is quantum theory consistent? From 1927 to 1930, Bohr and Einstein debated this question [1]. Both were familiar with the formalism, yet they hardly referred to it. They did not need to. With thought experiments, Einstein would argue that quantum theory is inconsistent; and Bohr would refute Einstein's arguments, one by one. After these refutations, Einstein conceded that quantum theory is consistent. So we can gain insight into quantum theory without the mathematical formalism. This chapter takes us to the climax of the Bohr-Einstein debate.

2.1 Why does the Color of the Light Change?

Let's visit a potter's studio. In the studio we see boxes of clay, jars of powdered glaze, spoons, brushes and rolling pins, and a potter's wheel; but what stands out is the kiln, sitting on metal posts, with its thick ceramic walls covered in metal. The huge, hot kiln dominates the studio. We peep into the hot kiln through a peephole in the door of the kiln. What do we see?

As the kiln's temperature reaches 1200 °C, the potter turns off the heating element. We then see the glow of matter heated to this temperature. The temperature of the kiln drops slowly, despite the peephole – roughly a degree per minute. So, to a good approximation, the kiln remains at thermal equilibrium. At 1200 °C (1473 K), the light we see through the peephole is orange mixed with yellow, and so bright it hurts the eyes. Almost everything in the kiln is the same color, though objects close to the peephole look darker than the background. In a few hours, the temperature of the kiln drops to 1000 °C (1273 K); the light is orange and less bright than before. We can see the outlines of some objects in the kiln. In a few more hours the temperature drops to 800 °C (1073 K) and the light is less intense; the color of the light is a mixture of orange and red. When, a few hours later, the temperature falls below 600 °C (873 K), we see only a dull red glow.

The correlation between temperature and the color of light is familiar and we take it for granted. We use such expressions as "red hot" and "white hot". A potter can judge the temperature of a kiln by the color of its glow. All the same, the correlation is mysterious. We understand why the intensity of the light changes: light has energy and the energy of the kiln decreases with the decreasing temperature. But why does the color of the light change?

Let us discuss in more detail *how* it changes. The electromagnetic radiation from a kiln is a mixture of light frequencies and other frequencies. The total energy in the radiation depends

Quantum Paradoxes: Quantum Theory for the Perplexed. Y. Aharonov and D. Rohrlich

ISBN 3-527-40391-4

on the size of the kiln, but the density of energy in the radiation does not; it depends only on the temperature of the kiln. Let $u(\nu, T)$ denote the density of energy in radiation of frequency ν coming from a kiln at absolute temperature T. That is, for small $d\nu$, the density of energy in radiation with frequencies between ν and $\nu + d\nu$ is $u(\nu, T)d\nu$. In 1860, Kirchhoff proved that $u(\nu, T)$ is the same for any *black body* in thermal equilibrium at temperature T. (By definition, a black body does not reflect radiation; but it emits radiation, so, despite the name, black bodies can have color. A kiln emits black-body radiation.) But Kirchhoff did not have sufficient data to determine $u(\nu, T)$. In 1896 Wien proposed a law to fit the data that had gradually accumulated:

$$u(\nu, T) = b\nu^3 e^{-a\nu/T} ,$$

where a and b are empirical constants. Then in 1900, when new data contradicted Wien's law, Planck proposed

$$u(\nu, T) = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{h\nu/kT} - 1} , \tag{2.1}$$

where k is Boltzmann's constant, c is the speed of light, and h is Planck's constant. (Planck's value for his constant was $h = 6.55 \times 10^{-27}$ erg sec; as of 2004, the accepted value is $h = [6.6260693 \pm 0.0000011] \times 10^{-27}$ erg sec. We note $\hbar \equiv h/2\pi \approx 1.054572 \times 10^{-27}$ erg sec.) Both Wien's law and Planck's law imply that the color of the light from a black body changes with its temperature, because the shape of $u(\nu, T)$ changes with T. And indeed the color changes, not only at temperatures that a kiln can reach, but also at higher temperatures. (See Fig. 2.1(a).)

We have defined the energy density $u(\nu, T)$ as a function of the frequency of radiation. We can just as well define energy density as a function of the wavelength λ of radiation. For small λ, let $u_\lambda(\lambda, T)d\lambda$ be the density of energy in radiation with wavelengths between λ and $\lambda + d\lambda$. For any two wavelengths $\lambda_1 = c/\nu_1$ and $\lambda_2 = c/\nu_2$, we must have

$$\int_{\nu_1}^{\nu_2} u(\nu, T)d\nu = \int_{\lambda_2}^{\lambda_1} u_\lambda(\lambda, T)d\lambda ,$$

so Eq. (2.1) implies

$$u_\lambda(\lambda, T) = \frac{8\pi hc}{\lambda^5} \frac{1}{e^{hc/\lambda kT} - 1} .$$

Figure 2.1(b) shows that the light from a kiln at 1473 K is mainly red, orange, yellow and green, mixing to a yellowish orange; at 1273 K the light is mainly red, orange and yellow; at 1073 K the light is a mixture of red and orange; and at 873 K the light is dark red. (See also Prob. 2.1.) But *why* does the color of the light change?

Can we derive $u(\nu, T)$? Boltzmann and Gibbs had already invented statistical mechanics when Planck proposed his law. A principle of statistical mechanics, the equipartition theorem, states that the average kinetic energy of a system with n degrees of freedom at temperature T is $nkT/2$. Between 1900 and 1905, Rayleigh, Einstein and Jeans applied the equipartition theorem to Maxwell's electromagnetism to obtain

$$u(\nu, T) = \frac{8\pi\nu^2}{c^3} kT . \tag{2.2}$$

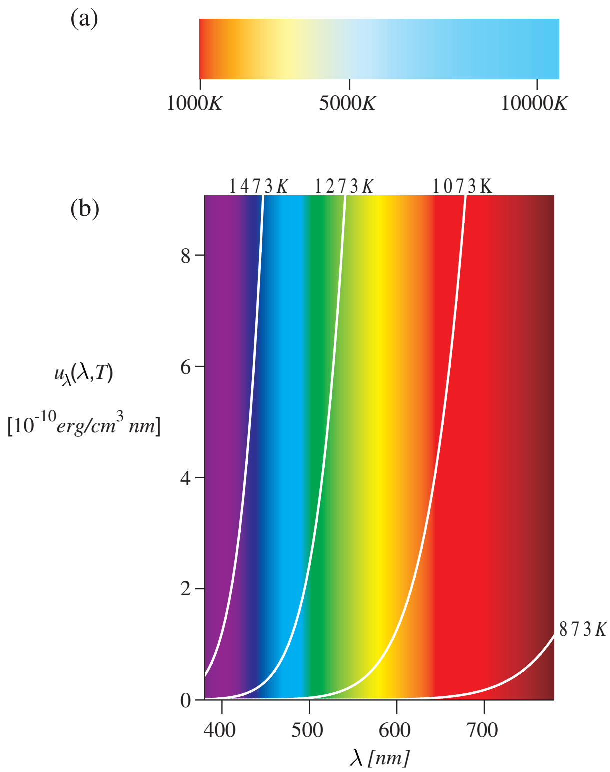

Figure 2.1: (a) Color of blackbody radiation as a function of temperature T. (b) Planck blackbody distribution $u_\lambda(\lambda, T)$ for temperatures 1473 K, 1273 K, 1073 K and 873 K, showing color of light as a function of wavelength λ. [We thank Dr. Dan Bruton for the two color spectra in this figure. Because of differences between light *emission* (from a computer screen) and *reflection* (from a printed page), the colors here differ slightly from those in his web site www.physics.sfasu.edu/astro/color.html.]

Each frequency ν of electromagnetic radiation corresponds to two degrees of freedom (two independent polarizations) and the density of frequencies is $4\pi\nu^2/c^3$. (See Prob. 2.2.) So Eq. (2.2) is simply kT times[1] the density of degrees of freedom, $8\pi\nu^2/c^3$. Note that Planck's law (unlike Wien's law) approaches Eq. (2.2) for $\nu \to 0$, but both laws contradict Eq. (2.2) for large ν.

Equation (2.2), in which T appears as an overall factor, implies that only the intensity of the light from a kiln changes with temperature, not the color. But Eq. (2.2) is wrong: if we integrate Eq. (2.2) to obtain the overall energy $U(T)$ in the radiation field,

$$U(T) = \int_0^\infty u(\nu, T) d\nu \ ,$$

[1]The factor kT includes the average of both kinetic and potential energy. These are equal for electromagnetic radiation as they are for harmonic oscillators.

we find that the integral diverges. Since statistical mechanics and electromagnetism together imply this divergence, statistical mechanics and electromagnetism together contain a contradiction. We must modify one or both of these two theories to make them compatible.

So why does the color of the light change? We can guess that it changes to avoid the divergence. The integral of Eq. (2.1) does *not* diverge. And Eq. (2.1) – Planck's law – implies that the color of the light changes with temperature. (See Prob. 2.3.)

2.2 Quanta

Planck's own derivation of Eq. (2.1) was "an act of desperation ... I had to obtain a positive result, under any circumstance and at whatever cost", as he put it [2]. Oddly, Planck was not aware of Eq. (2.2); but he was aware that he could not derive Eq. (2.1) in any reasonable way. To derive his law, he assumed that matter is composed of harmonic oscillators that exchange energy with the electromagnetic field. This assumption was reasonable enough. He also assumed that a harmonic oscillator of frequency ν could not exchange energy in arbitrary amounts, but only in *quanta* of energy $h\nu$. This assumption was completely unreasonable. According to classical theory, h should vanish; and as h vanishes, Planck's law reduces to Eq. (2.2).

Five years later, Einstein extended Planck's assumption. He assumed that electromagnetic radiation itself consists of quanta; radiation of frequency ν consists of quanta of energy E with

$$E = h\nu \ . \tag{2.3}$$

Einstein applied Eq. (2.3) to the photoelectric effect. Metals exposed to ultraviolet light emit electrons. The energy of the emitted electrons depends on the frequency, but not on the intensity, of the light. Einstein predicted a linear relation between the light frequency and the energy of the electrons, with a slope, independent of the type of metal, equal to Planck's constant. Experiments verified these predictions by 1916. Yet almost no one accepted Einstein's hypothesis of light quanta [3]. Light is a wave; how could light quanta produce interference?

Then in 1923, Compton showed that light, scattering off electrons at rest, imparts momentum in an amount that depends on the wavelength of the light, but not on its intensity. He found a clear relationship between θ, the angle through which the light scattered, and the change in its wavelength:

$$\lambda_f - \lambda_i = \frac{h}{mc}(1 - \cos\theta) \ ,$$

where λ_f and λ_i are the final and initial wavelengths of the light, respectively, and m is the mass of the electron. The Compton effect strongly suggests that light quanta – *photons* – of wavelength λ carry momentum h/λ as well as energy $h\nu$, for then the relationship follows from conservation of energy and momentum. (See Prob. 2.4.) In the same year, de Broglie proposed that if light waves could behave like particles, then particles could behave like waves. Four years later, Davisson and Germer observed electron diffraction and confirmed de Broglie's relation between the momentum p and the wavelength λ of a particle:

$$p = h/\lambda \ . \tag{2.4}$$

Quanta had arrived.

Section 1.4 notes that relativity theory resolves a paradox: electromagnetism and Newton's mechanics are incompatible. The theory of relativity resolves this paradox by modifying Newton's mechanics. In retrospect, we see that quantum theory, too, resolves a paradox. Statistical mechanics and electromagnetism are incompatible; together, they imply the Rayleigh–Einstein–Jeans law, Eq. (2.2), and an infinite energy density $U(T)$ for electromagnetic radiation. This is a paradox of the third class, the class of contradictions. (See Chap. 1.) Quantum theory resolves the paradox by modifying electromagnetism: electromagnetic radiation of frequency ν cannot carry energy in arbitrary amounts, but only in quanta of energy $h\nu$. Together, statistical mechanics and the modified electromagnetism imply the Planck law, Eq. (2.1), and a finite $U(T)$. (See Prob. 2.5.)

2.3 Uncertainty Relations

Quanta behave like waves and like particles. Are they waves or particles? Whatever they are, quanta confront us with a paradox each time we make a measurement. If quanta behave like waves, how can we measure their position? If they behave like particles, how can we measure their wavelength? We can live with the paradox, but it implies fundamental limits to what we measure. Here we derive these limits informally; for a formal derivation, see Prob. 3.10 and Sects. 5.3 and 7.3.

Consider a measurement with a microscope. A light microscope can resolve features of small objects, up to a limit. The limit depends on the wavelength of the light. The smallest separation Δx that a lens can resolve in its object plane (the x-axis in Fig. 2.2) is approximately

$$\Delta x \approx \lambda/2\sin\theta \ ,$$

where λ is the wavelength of the light and θ is half the angle subtended by the lens at the object. (See Prob. 2.6.) So if we want to determine the position of a small object with an accuracy Δx, we need light of wavelength $\lambda \leq 2(\Delta x)\sin\theta$. In both classical and quantum physics, we *have* light of such short wavelengths. But in quantum physics, short wavelengths correspond to quanta carrying high momenta, as Eq. (2.4) shows. A high momentum photon scatters off the measured object and alters its momentum. Suppose we illuminate the object from the side with light of wavelength $\lambda = 2(\Delta x)\sin\theta$. The light consists of photons of momentum $p^\gamma = h/\lambda = h/2(\Delta x)\sin\theta$ in the x-direction. When a photon scatters off the object we do not know in which direction it scatters, only that it reaches the lens (if it is at all relevant to the measurement); thus all that we know about its final momentum in the x-direction is that it lies between $-p^\gamma\sin\theta$ and $p^\gamma\sin\theta$, i.e. between $-h/2\Delta x$ and $h/2\Delta x$. The photon alters the momentum of the object by this uncertain amount, hence the measurement of the object's position along the x-axis leaves us uncertain about its momentum in the x-direction; the uncertainty Δp_x in its momentum is at least $\Delta p_x \geq h/\Delta x$. In particular, we cannot rely on a prior measurement of momentum for predicting the future position of the object. This is the meaning of the Heisenberg uncertainty relation [4]:

$$\Delta x \Delta p_x \geq h \ . \tag{2.5}$$

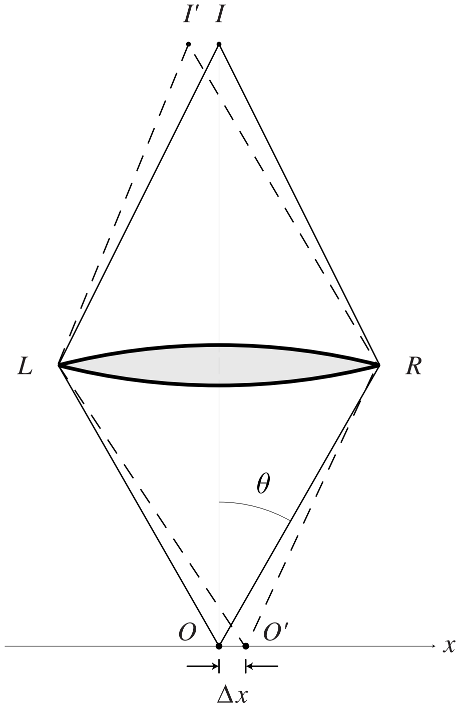

Figure 2.2: Two points O and O' with separation Δx; their respective images, each the maximum of a diffraction pattern, are I and I'.

Equation (2.5) is revolutionary. Consider, for example, an atom of hydrogen. Its radius is roughly the Bohr radius $a_0 = 0.53\ \mathring{A}$ (that is, 5.3 $\times 10^{-9}$ cm); its ionization energy is 13.6 electron volts (eV). Suppose we measure the position of the electron to better than the Bohr radius, i.e. $\Delta x < 5.3 \times 10^{-9}$ cm. According to Eq. (2.5), our position measurement entails uncertainty in the electron's momentum of at least

$$\Delta p > h/\Delta x \approx 1.25 \times 10^{-18} \mathrm{g\ cm/sec}\ .$$

We might estimate the kinetic energy of the electron after the position measurement to be

$$(\Delta p/2)^2/2m \approx (6 \times 10^{-19} \mathrm{g\ cm/sec})^2/2 \times (9.1 \times 10^{-28} \mathrm{g}) \\ \approx 2 \times 10^{-10} \mathrm{erg} \approx 130 \mathrm{eV}\ , \tag{2.6}$$

where m$\approx$ 9.1×10^{-28} g is the mass of the electron. This kinetic energy is greater than the ionization energy, so the attempt to locate the electron within the atom ionizes the atom! Actually, we have overestimated the kinetic energy,[2] but any attempt to localize the electron to a well defined orbit within the hydrogen atom will indeed ionize the atom.

We obtained Eq. (2.5) from an experiment with a microscope, but Eq. (2.5) holds for any measurement of position and momentum. We always find that conditions for a precise measurement of x conflict with conditions for a precise measurement of p. The conflict illustrates

[2] A good estimate of the kinetic energy is $(\hbar/\Delta x)^2/2m \approx 14$ eV.

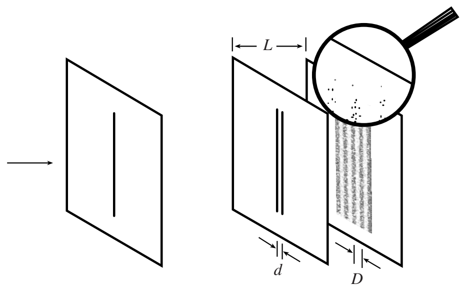

Figure 2.3: Two-slit interference experiment. Electrons enter from the left in the direction of the arrow. Magnification shows dots making up the interference pattern.

Bohr's principle of *complementarity*: measurements of canonically conjugate variables (such as x and p) impose conflicting conditions. The more an experiment fulfills the conditions for measuring one variable, the less it fulfills the conditions for measuring the conjugate variable. When we quantify the complementarity between the measurements, we obtain Eq. (2.5) (and analogous uncertainty relations for other pairs of conjugate variables).

Complementarity allows us to live with paradox, but the paradox remains. Particles and waves are complementary pictures of quanta. Each picture contains a part of the truth; but there is no picture uniting the wave and particle pictures. Indeed, they contradict each other. Consider a beam of electrons impinging on three screens. (See Fig. 2.3.) The first screen has only one slit. The second screen has two slits, separated by a distance d. The distance between the two screens is much larger than d, so waves passing through the first screen and arriving at the two slits of the second screen have effectively parallel wave vectors. The waves passing through the two slits interfere, producing a pattern of alternating light and dark bands on the third screen, a distance L from the second. The spacing between adjacent dark bands is D. This is the familiar phenomenon of wave interference, with

$$D \approx \lambda L/d \,, \tag{2.7}$$

if the wavelength is λ. What is new is that the electrons are not simply waves. They also behave like particles. If the beam intensity drops until only one electron passes through the apparatus at a time, the pattern of light and dark bands still appears. The light and dark bands emerge from marks that appear one by one on the screen, even when the time interval between successive marks is longer than the time of flight of an electron through the apparatus [5].

2.4 The Clock-in-the-Box Paradox

The double-slit experiment figured in the Bohr-Einstein debate on whether quantum theory is consistent. Einstein saw in it a paradox. Suppose we prepare the middle screen with no transverse momentum, and measure its transverse momentum after an electron passes through it. (See Fig. 2.4(a).) By measuring the recoil of the screen after the electron passes, we can infer through which slit it passed. Let us denote the electron's final transverse momentum by $p_\perp^{(L)}$ if the electron passes through the left slit and by $p_\perp^{(R)}$ if it passes through the right slit. (See Fig. 2.4(b).) If the electron passes through the left slit and arrives at point $\mathcal{P}$, the middle screen must acquire momentum $-p_\perp^{(L)}$ to conserve momentum; if it passes through the right

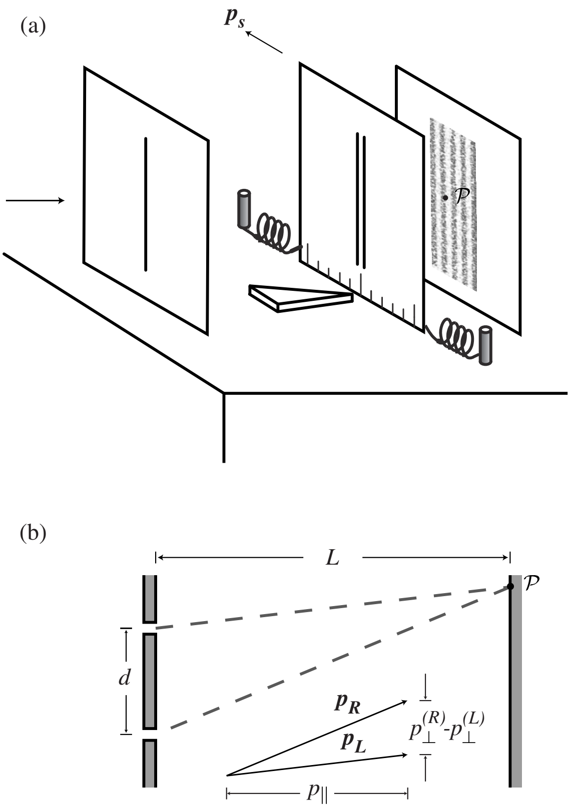

Figure 2.4: (a) The two-slit interference experiment of Fig. 2.2 adapted for measuring the transverse momentum of the middle screen. (b) The second and third screens seen from above, with interfering electron paths and corresponding momenta.

slit on its way to $\mathcal{P}$, the middle screen acquires momentum $-p_\perp^{(R)}$. Thus we can determine through which slit the particle passed by measuring the final momentum of the middle screen. How can there be an interference pattern? This is a paradox of the first class, an error. Bohr resolved the paradox by applying the uncertainty relations consistently. If we measure the momentum p_s of the screen with an accuracy Δp_s, then any simultaneous measurement of the position x_s of the screen entails an uncertainty Δx_s such that

$$\Delta x_s \geq h/\Delta p_s \ . \tag{2.8}$$

How well do we need to measure p_s? We want to detect whether a particle that arrives at $\mathcal{P}$ came via the left slit or the right one. In order to determine through which slit the electron passes, we must measure p_s to accuracy Δp_s better than $p_\perp^{(R)} - p_\perp^{(L)} = |\mathbf{p}^{(R)} - \mathbf{p}^{(L)}|$. From similarity of triangles, $p_\perp^{(R)} - p_\perp^{(L1)}$ divided by the electron's longitudinal momentum $p_{||}$ is equal to d/L. The longitudinal momentum $p_{||}$, according to de Broglie, is h/λ (assuming $p_{||}$ large compared to the transverse momentum). Thus

$$\Delta p_s < \frac{d}{L}(h/\lambda) \ . \tag{2.9}$$